Localisation for non-monotone Schrödinger operators

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Abstract

We study localisation effects of strong disorder on the spectral and dynamical properties of (matrix and scalar) Schrödinger operators with non-monotone random potentials, on the d-dimensional lattice. Our results include dynamical localisation, i.e. exponentially decaying bounds on the transition amplitude in the mean. They are derived through the study of fractional moments of the resolvent, which are finite due to resonance-diffusing effects of the disorder. One of the byproducts of the analysis is a nearly optimal Wegner estimate. A particular example of the class of systems covered by our results is the discrete alloy-type Anderson model.

1 Introduction

1.1 Random Schrödinger operators

The prototypical model for the study of localisation properties of quantum states of single electrons in disordered solids is the Anderson Hamiltonian H_A, which

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acts on $\ell^2(\mathbb{Z}^d)$ by

$$(H_A\psi)(\mathbf{n}) = v(\mathbf{n})\psi(\mathbf{n}) + g^{-1} \sum_{\mathbf{m} \text{ adjacent to } \mathbf{n}} \psi(\mathbf{n}) \ ,$$

where the entries $v(\mathbf{n})$ of the potential are random and independent.

The basic phenomenon, named Anderson localisation after the physicist P. W. Anderson, is that disorder can cause localisation of electron states, which manifests itself in time evolution (non-spreading of wave packets), (vanishing of) conductivity in response to electric field, Hall currents in the presence of both magnetic and electric field, and statistics of the spacing between nearby energy levels. The first property implies spectral localisation, i.e. the spectral measure of H_A is almost surely pure point, and almost sure exponential decay of eigenfunctions.

These properties are known to hold for H_A in each of the following cases: 1) high disorder (the coupling constant g is large), 2) extreme energies, 3) weak disorder away from the spectrum of the unperturbed operator, and 4) one dimension, $d = 1$.

Historically, the first proof of spectral localisation was given by Goldsheid, Molchanov and Pastur [12], for a one-dimensional continuous random Schrödinger operator.

In higher dimension, the absence of diffusion was first established in 1983 by Fröhlich and Spencer [9] using multi-scale analysis. Their approach has led to a multitude of results on localisation for a wide range of problems. The reader is referred to the monograph of Stollmann [17] or the recent lecture notes of W. Kirsch [14] for a review of the history of the subject and a gentle introduction to the multi-scale analysis — which is not used here.

One of the ingredients of multi-scale analysis is the regularity of the *integrated density of states*, the (distribution function of the) average of the spectral measure over the randomness.

Ten years later Aizenman and Molchanov [2] introduced an alternative method for the proof of localisation, known as the fractional moment method, which has also found numerous applications. In particular, in [1], Aizenman introduced the notion of eigenfunction correlator, which, combined with the fractional moment method, allowed him to give the first proof of dynamical localisation. We refer to the lecture notes of Stolz [19] and Aizenman and Warzel [3] for a survey of subsequent developments.

In the fractional moment method, an a priori estimate on the diagonal elements of the resolvent $(H_A - \lambda)^{-1}$ plays a key rôle in the underlying analysis.

In many situations, regularity of the integrated density of states follows from the regularity of the distribution of the potential. This was first proved by Wegner [20], therefore regularity estimates on the density of states are called Wegner estimates. An essential ingredient in his argument is the monotone dependence of the spectrum of H_A on the random variables $v(\mathbf{n})$. A modification of this argument was applied by Aizenman and Molchanov to give an a priori bound on the average of $|(H_A - \lambda - i0)^{-1}(x,x)|^s$.

The monotone dependence of the spectrum of the random variables is also used in the fractional moment's proof of dynamical localisation (via a variant of spectral averaging).

Recently, several challenging problems (listed below) arose in different contexts, in which the dependence of the spectrum on the random variables is not monotone. In this paper, we develop a strategy to prove localisation which is applicable to some of these models.

One close relative of the original Anderson Hamiltonian H_A is the random *alloy-type* model, in which the potential $V(\mathbf{n})$ at a site $\mathbf{n} \in \mathbb{Z}^d$ is obtained from independent random variables $v(\mathbf{m})$ via the formula

$$V(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbf{\Gamma}} a_{\mathbf{n}-\mathbf{k}} v(\mathbf{k}) \,, \tag{1.1}$$

where the index $\mathbf{k}$ takes values in some sub-lattice $\mathbf{\Gamma}$ of $\mathbb{Z}^d$. If all the coefficients $a_{\mathbf{k}}$ have the same sign, the system is monotone, since the dependence of the spectrum on V is monotone. Localisation in such systems is well understood by now, even in the continuum setting. The existing technology is however not well suited to the non-monotone case, i.e. when $a_{\mathbf{k}}$ are not all of the same sign. Mathematically, the problem becomes especially acute when $\sum a_{\mathbf{k}} = 0$.

There is no physically compelling reason for a random tight binding alloy model to be monotone, and the natural question is whether Anderson localisation still holds if one breaks the monotony.

Non-monotone models also naturally appear in the class of *block operators*. In one such model, introduced by Fröhlich, and studied by Bourgain in [4], the matrix-valued potential is given by $V(\mathbf{n}) = U(\mathbf{n})^* A U(\mathbf{n})$, where A is a fixed self-adjoint $r \times r$ matrix, and $U(\mathbf{n})$ are independently chosen according to the Haar measure on $SU(r)$. Bourgain proved a volume-dependent Wegner estimate and Anderson localisation near the edges of the spectrum using methods from complex analysis.

Another class of non-monotone random block operators, associated with the Bogoliubov – de Gennes symmetry classes, was studied by Kirsch, Metzger, and Müller in [15] and by Gebert and Müller in [10]. The former paper establishes the Wegner estimate for such random operators in a neighborhood of the inner band edges, by recovering some form of the monotonicity. The latter paper uses the bootstrap multi-scale analysis to prove dynamical localization in the same regime.

The original motivation for this work was to study a problem suggested by Tom Spencer, in which the matrix-valued potential is of the form

$$V(\mathbf{n}) = \begin{pmatrix} v(\mathbf{n}) & a \\ a & -v(\mathbf{n}) \end{pmatrix} ,$$

where the variables $v(\mathbf{n})$ are independent and identically distributed. If the distribution of $v(\mathbf{n})$ has bounded density, the eigenvalue distribution of a single $V(\mathbf{n})$ is $1/2$-Hölder; this is optimal, since the density of the eigenvalue distribution diverges as $|\lambda \mp a|^{-1/2}$ at the energies $\pm a$. Spencer conjectured that the integrated density of states for the full Hamiltonian is also at least $1/2$-Hölder.

Electromagnetic Schrödinger operators with random magnetic field, the random displacement model, random block operators and Laplace-Beltrami operators with random metrics are other examples of systems with non-monotone parameter dependence which were intensively studied recently.

We refer to the paper of Elgart, Krüger, Tautenhahn, and Veselić [7] for a survey of recent results on the sign indefinite alloy-type models with $\mathbf{\Gamma} = \mathbb{Z}^d$ and the bibliography pertaining to some of the models mentioned in the previous paragraph.

Summary of results. For the last few years there has been a continuous effort to bring the understanding of models with non-monotone dependence on the randomness to a same level as the one for monotone models. In this paper, we present a method to prove Anderson localisation and a Wegner estimate for several non-monotone models, achieving this goal. Theorem 1.1 and its corollaries pertain to a class of models with matrix-valued potentials; when applied to Spencer's model, it shows that, under some assumptions on the distribution of $v(\mathbf{n})$, the integrated density of states is $(1/2 - \epsilon)$-Hölder for any $\epsilon > 0$, at large coupling g. Unfortunately, Theorem 1.1 does not directly apply to Fröhlich's model.

Theorem 1.2 and its corollaries establish Anderson localisation and a Wegner estimate for the alloy-type model (1.1), in the case that a is finitely supported.

Our argument can be viewed as a further augmentation of the fractional moment method of Aizenman–Molchanov [2]. In particular, Proposition 2.1 is a modification of [2, (2.25)], whereas Proposition 3.1 is a version of the decoupling estimates [2, Lemmata 2.3,3.1]. The innovation of this work is that we do not rely on an a priori estimate on the moments of diagonal resolvent elements; instead, we prove such an estimate in parallel with localisation. We also propose an argument which allows to deduce dynamical localisation directly from the resolvent estimates, and which works in the non-monotone setting as well as in the monotone one.[1]

Relation to some past and present works. For the one dimensional continuum alloy-type random models the proof of the complete Anderson localization was first given by Stolz [18].

Outside the spectrum of the unperturbed operator (corresponding to the random potential being switched off) one can obtain Lipschitz regularity of the integrated density of states by reducing the problem to the monotone case. The optimal Wegner estimate in this case was established by Combes, Hislop, and Klopp [5] (in the continuum, but their argument is equally applicable in the discrete setting). This input can be used to prove Anderson localisation in the regimes of extreme energies and weak disorder away from the spectrum of the unperturbed operator (for the latter regime for the continuum models this result goes back to Klopp [16]).

Our work covers the remaining perturbative setting, namely the high disorder regime, where we prove complete localisation.

Recently, Bourgain (private communication) devised a different approach that allows to prove s-regularity of the density of states for a wide class of non-monotone models which includes Fröhlich's model, as well as some of the models we consider in this note.

1.2 Notation and statement of results

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with degree at most κ; the set of vertices (sites) $\mathcal{V}$ may be either finite or countable. The main example is the lattice $\mathcal{G} = \mathbb{Z}^d$ (where $\kappa = 2d$), however, the greater generality does not require additional effort here.

[1] After this work was completed, we have learned from Victor Chulaevsky that a similar argument was used earlier by Germinet and Klein in the context of the multi-scale analysis, see [11, Theorem 1, Corollary 1].

For $x, y \in \mathcal{V}$, denote by $\mathrm{dist}(x, y)$ the length of the shortest path connecting x to y; when $\mathcal{G} = \mathtt{Z}^d$,

$$\mathrm{dist}(x, y) = \|x - y\|_1 \, .$$

Let

$$v : \Omega \times \mathcal{V} \longrightarrow \mathtt{R}$$

be a collection of independent, identically distributed random variables, where $(\Omega, \mathtt{P})$ is a probability space, and we assume that the distribution μ of every $v(x)$

A1 is α-regular for some $\alpha > 0$, meaning that $\mu[t - \epsilon, t + \epsilon] \leq C_{\mathbf{A1}} \epsilon^\alpha$ for any $\epsilon > 0$ and $t \in \mathtt{R}$;

A2 has a finite q-moment for some $q > 0$, meaning that $\int |x|^q d\mu(x) \leq C_{\mathbf{A2}}$.

For example, the Gaussian distribution and the uniform distribution on a finite interval satisfy **A1** with $\alpha = 1$ and **A2** with any $q > 0$.

We shall denote the expectation by $\langle \cdot \rangle$ and the expectation over the distribution of one $v(x)$ by $\langle \cdot \rangle_{v(x)}$.

In the electron gas approximation the system of electrons in a crystal is modeled by a gas of Fermions moving on a lattice. The excitations of the system are described by an effective one-body Hamiltonian H, which consists of a short-range hopping term and a local (single site) potential. Each site x of the lattice will be assumed to have k internal degrees of freedom.

Single site (matrix) potential: For any $x \in \mathcal{V}$, define a Hermitian matrix

$$V(x) = v(x)A(x) + B(x) \, ,$$

where the Hermitian $k \times k$ matrices $A(x)$ and $B(x)$ satisfy

B1 $\|A(x)\|, \|A(x)^{-1}\| \leq C_{\mathbf{B1}}$;

B2 $\|B(x)\| \leq C_{\mathbf{B2}}$.

Hopping: For every ordered pair $(x, y) \in \mathcal{V} \times \mathcal{V}$ of adjacent sites (i.e. $(x, y) \in \mathcal{E}$) we introduce a $k \times k$ matrix (*kernel*) $K(x, y)$ so that

B3 $K(y, x) = K(x, y)^*$ and $\|K(x, y)\| \leq C_{\mathbf{B3}}$.

We are now in position to introduce our one-particle Hamiltonian. Namely, let H be a random operator acting on $\ell^2(\mathcal{V}) \otimes \mathbb{C}^k$ (the space of square-summable functions $\psi : \mathcal{V} \to \mathbb{C}^k$)

$$(H\psi)(x) = V(x)\psi(x) + g^{-1} \sum_{y \sim x} K(x,y)\psi(y) \,, \tag{1.2}$$

where $g > 0$ is a coupling constant, and the sum is over all $y \in \mathcal{V}$ such that $(x,y) \in \mathcal{E}$.

Let $G_\lambda = (H - \lambda)^{-1}$ be the resolvent of H, $\lambda \notin \mathbb{R}$. It is known that the limit $G_{\lambda+i0} = \lim_{\epsilon \to +0} G_{\lambda+i\epsilon}$ exists for almost every $\lambda \in \mathbb{R}$. In the following, $\langle \|G_{\lambda+i0}(x,y)\|^s \rangle$ can a priori be formally interpreted as

$$\lim_{\epsilon \to +0} \langle \|G_{\lambda+i\epsilon}(x,y)\|^s \rangle \,;$$

a posteriori, $G_{\lambda+i0}$ is finite almost surely, and

$$\lim_{\epsilon \to +0} \langle \|G_{\lambda+i\epsilon}(x,y)\|^s \rangle = \langle \| \lim_{\epsilon \to +0} G_{\lambda+i\epsilon}(x,y)\|^s \rangle \,.$$

Theorem 1.1. *Let $0 < s \le \frac{\alpha q}{2k\alpha + kq}$. There exists $C > 0$ that may depend on α, q, $C_{\mathbf{A1}}$–$C_{\mathbf{B3}}$ and s such that for any $\lambda \in \mathbb{R}$ and any $g \ge C\kappa^{1/s}/(1+|\lambda|)$*

$$\langle \|G_{\lambda+i0}(x,y)\|^s \rangle \le \frac{C}{(1+|\lambda|)^s} \left(\frac{C\kappa}{g^s (1+|\lambda|)^s} \right)^{\mathrm{dist}(x,y)} \,.$$

Let us state some corollaries for the homogeneous setting; for simplicity, assume that $\mathcal{G} = \mathbb{Z}^d$ (we denote the vertices of $\mathbb{Z}^d$ by $\mathbf{m}, \mathbf{n}, \cdots$) We also assume that

C $A(\mathbf{m}) \equiv A$, $B(\mathbf{m}) \equiv B$, $K(\mathbf{m}, \mathbf{n}) \equiv K(\mathbf{m} - \mathbf{n})$.

The density of states ρ is defined as the average of the spectral measure corresponding to H:

$$\int f(\lambda) d\rho(\lambda) = \frac{1}{k} \mathrm{tr} \, \langle f(H)(\mathbf{n}, \mathbf{n}) \rangle \,,$$

where tr stands for the trace. The integrated density of states is the distribution function $\lambda \mapsto \rho(-\infty, \lambda]$ of ρ.

The assumption **C** guarantees that these definitions do not depend on the choice of the vertex $\mathbf{n} \in \mathbb{Z}^d$.

Theorem 1.1 implies the following Wegner-type estimate:

Corollary 1.1.1. *Assume* **C**. *If $g \geq Cd^{1/s}/(1+|\lambda|)$, then the integrated density of states is locally s-Hölder at λ for*

$$s = \frac{\alpha q}{2k\alpha + kq} = \frac{\alpha}{k\left(1 + \frac{2\alpha}{q}\right)} ,$$

uniformly in $g \to \infty$:

$$\rho[\lambda - \epsilon, \lambda + \epsilon] \leq C(1+|\lambda|)^{-s}\epsilon^s .$$

In particular, for any distribution with bounded density and finite moments the integrated density of states is $1/(k+\epsilon)$-Hölder for any $\epsilon > 0$.

Next, Theorem 1.1 implies dynamical and spectral Anderson localisation:

Corollary 1.1.2. *Assume* **C**. *Let I be a finite interval of energies, and let*

$$g \geq \frac{Cd^{1/s}}{1 + \min_{\lambda \in I} |\lambda|} .$$

Then, for any $\mathbf{m} \neq \mathbf{n} \in \mathbb{Z}^d$,

$$\langle \sup_{t \geq 0} \left| e^{itH_I}(\mathbf{m}, \mathbf{n}) \right| \rangle \leq C \mathrm{dist}(\mathbf{m}, \mathbf{n})^{2d} \left(\frac{Cd}{g^s (1+|\lambda|)^s} \right)^{\frac{s \, \mathrm{dist}(\mathbf{m}, \mathbf{n})}{8}} , \qquad (1.3)$$

where $H_I = P_I H P_I$, P_I is the spectral projector corresponding to I. Therefore the spectrum of H in I is almost surely pure point.

The first part of the last corollary follows from Theorem 1.1, (4.1), and Theorem 4.2. The "therefore" part follows from the summability of the right-hand side of (1.3) via the Kunz–Souillard theorem [6, Theorem 9.21].

1.3 Extensions

Alloy-type models. Consider the operator H with potential (1.1) acting on $\ell^2(\mathbb{Z}^d)$. Let $\mathcal{B}_{\mathbf{n}}$ be the set of $v(\mathbf{m})$ for which $a_{\mathbf{n}-\mathbf{m}} \neq 0$. We will impose the following assumptions on the random potential:

1. the set $\mathcal{B}_{\mathbf{n}}$ is non empty for all $\mathbf{n}$;
2. the cardinality $k = \#\{\mathbf{m} \,|\, a_{\mathbf{m}} \neq 0\} < \infty$;

3. the distribution of $v(\mathbf{m})$ satisfies **A1** and **A2**.

Theorem 1.2. *Let $0 < s < \frac{\alpha q}{2k\alpha + kq}$. There exists $C > 0$ such that for any $\lambda \in \mathtt{R}$ and any $g \geq Cd^{1/s}/(1+|\lambda|)$*

$$\langle |G_{\lambda+i0}(\mathbf{m},\mathbf{n})|^s \rangle \leq \frac{C}{(1+|\lambda|)^s} \left(\frac{Cd}{g^s(1+|\lambda|)^s} \right)^{\mathrm{dist}(\mathbf{m},\mathbf{n})} .$$

Similarly to Corollaries 1.1.1,1.1.2, one can deduce a Wegner estimate and Anderson localisation.

Relaxing the covering condition. The assumption $\mathbf{B1}$ is usually referred to as a covering condition. In our analysis, it enters in the proof of Lemma 2.2. In particular, all our results are still valid (albeit with the less sharp bound on the underlying localisation length) if one replaces $\mathbf{B1}$ with

B1′ $\langle \|(V(y)-\lambda)^{-1}\|^s \rangle_{v(y)} \leq Cg^\alpha$, with $\alpha < s$.

For a fixed non zero matrix $A(y)$ and a generic matrix $B(y)$ the estimate $\mathbf{B1}'$ holds true for g large enough. For instance $\mathbf{B1}'$ is applicable (with $\alpha = 0$) for

$$A(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; ; \quad B(y) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} .$$

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2 Proof of theorems

In the proof of Theorem 1.1, we assume that the graph $\mathcal{G}$ is finite and $\lambda \notin \mathtt{R}$. The estimates will be uniform in $\#\mathcal{V} \to \infty$ ($\#$ denotes cardinality) and $\mathrm{Im}\,\lambda \to 0$,

therefore the statement for infinite graphs and real λ can be deduced as follows. First, an infinite graph can be approximated by its finite pieces; the matrix elements of the resolvent corresponding to the finite pieces converge to the matrix elements of the resolvent corresponding to the infinite graph, yielding the same estimate for $\lambda \notin \mathrm{R}$. Then one can let $\mathrm{Im}\,\lambda$ go to zero.

The proof of Lemma 2.2 below will be postponed until Section 3.

Proposition 2.1. *For any $s \leq \frac{\alpha q}{2k\alpha+kq}$ there exists $C > 0$ (depending on s and the constants in the assumptions) such that for any $\lambda \notin \mathrm{R}$*

$$\langle \|G_\lambda(x,y)\|^s \rangle \leq \frac{C}{2(1+|\lambda|)^s} \left\{ g^{-s} \sum_{z\sim y} \langle \|G_\lambda(x,z)\|^s \rangle + \delta_{xy} \right\} ,$$

where

$$\delta_{xy} = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

is the Kronecker δ.

Proof. By definition of G_λ,

$$G_\lambda(x,y)(V(y)-\lambda) = -g^{-1} \sum_{z\sim y} G_\lambda(x,z) K(z,y) + \delta_{xy} .$$

Therefore

$$\langle \|G_\lambda(x,y)(V(y)-\lambda)\|^s \rangle \leq C^s_{\mathbf{B3}} g^{-s} \sum_{z\sim y} \langle \|G_\lambda(x,z)\|^s \rangle + \delta_{xy} .$$

Lemma 2.2. *For $s \leq \frac{\alpha q}{2k\alpha+kq}$, there exists $\hat{C}$ (depending on s and the constants in the assumptions) such that*

$$\langle \|G_\lambda(x,y)(V(y)-\lambda)\|^s \rangle \geq \hat{C}^{-1} \langle \|G_\lambda(x,y)\|^s \rangle \, (1+|\lambda|)^s .$$

The proposition follows. □

Corollary 2.2.1. *For any $s \leq \frac{\alpha q}{2k\alpha+kq}$, we have*

$$\max_y \langle \|G_\lambda(x,y)\|^s \rangle = \langle \|G_\lambda(x,x)\|^s \rangle ,$$

provided $g^s \geq C\kappa/(1+|\lambda|)^s$.

Proof. Suppose the maximum M is attained at $y \neq x$. Then

$$M = \langle \|G_\lambda(x,y)\|^s \rangle \leq \frac{C}{2g^s(1+|\lambda|)^s} \sum_{z \sim y} \langle \|G_\lambda(x,z)\|^s \rangle$$
$$\leq \frac{C\kappa M}{2g^s(1+|\lambda|)^s} \leq \frac{CM}{2C} = \frac{M}{2} ,$$

a contradiction. □

Corollary 2.2.2. *For any $s \leq \frac{\alpha q}{2k\alpha + kq}$ and $g^s \geq C\kappa/(1+|\lambda|)^s$*

$$\langle \|G_\lambda(x,x)\|^s \rangle \leq \frac{C}{(1+|\lambda|)^s} .$$

Proof. By Proposition 2.1 with $y = x$ and Corollary 2.2.1,

$$\langle \|G_\lambda(x,x)\|^s \rangle \leq \frac{C}{2(1+|\lambda|)^s} \left\{ g^{-s}\kappa \langle \|G_\lambda(x,x)\|^s \rangle + 1 \right\}$$
$$\leq \frac{1}{2} \langle \|G_\lambda(x,x)\|^s \rangle + \frac{C}{2(1+|\lambda|)^s} ,$$

therefore

$$\langle \|G_\lambda(x,x)\|^s \rangle \leq \frac{C}{(1+|\lambda|)^s} .$$

□

Proof of Theorem 1.1. For $x = y$ the inequality follows from Corollary 2.2.2. For $x \neq y$ apply Proposition 2.1 dist(x,y) times, and then use Corollary 2.2.1 and Corollary 2.2.2 to estimate every term. □

Proof of Theorem 1.2. The proof follows that of Theorem 1.1. The main modification (apart from replacing $\|\cdot\|$ with $|\cdot|$) appears in Lemma 2.2, which has to be replaced with

Lemma 2.3. *For $s \leq \frac{\alpha q}{2k\alpha + kq}$, there exists $\hat{C}$ such that*

$$\langle |G_\lambda(\mathbf{m},\mathbf{n})|^s |V(\mathbf{n}) - \lambda|^s \rangle \geq \hat{C}^{-1} \langle |G_\lambda(\mathbf{m},\mathbf{n})|^s \rangle (1+|\lambda|)^s .$$

The proof is provided at the end of Section 3.

□

3 Estimates on ratios of polynomials

Lemma 2.2 will follow from

Proposition 3.1. *Let μ be a probability measure satisfying the assumptions* **A1**, **A2**. *Let $a_1, \cdots, a_l, b_1, \cdots, b_m \in \mathbb{C}$, and let $s, r > 0$ be such that $rm < \alpha$ and $q \geq (sl + rm)\frac{\alpha}{\alpha - rm}$. Then*

$$\int \frac{\prod_{j=1}^{l} |v - a_j|^s}{\prod_{i=1}^{m} |v - b_i|^r} \, d\mu(v) \asymp \frac{\prod_{j=1}^{l} (1 + |a_j|)^s}{\prod_{i=1}^{m} (1 + |b_i|)^r} \ ,$$

where the "$\asymp$" sign means that LHS $\leq C$ RHS $\leq C'$ LHS, and the numbers $C, C' > 0$ may depend on α, q, $C_{\mathbf{A1}}$, $C_{\mathbf{A2}}$, l, m, r, and s, but not on a_j and b_i.

Proof of Lemma 2.2. First let us show that the statement holds for very small $s > 0$; then we shall extend it to all $s \leq \frac{\alpha q}{4\alpha + 2q}$. We shall consider the (slightly more complicated) case $x \neq y$.

For s sufficiently small,

$$\begin{aligned} &\langle \|G_\lambda(x,y)\|^s \rangle_{v(y)} \\ &\quad \leq \langle \|G_\lambda(x,y)(V(y) - \lambda)\|^s \|(V(y) - \lambda)^{-1}\|^s \rangle_{v(y)} \\ &\quad \leq \langle \|G_\lambda(x,y)(V(y) - \lambda)\|^{2s} \rangle_{v(y)}^{1/2} \langle \|(V(y) - \lambda)^{-1}\|^{2s} \rangle_{v(y)}^{1/2} \ . \end{aligned} \tag{3.1}$$

By the Schur–Banachiewicz formula for the inverse of a block matrix[2],

$$\begin{pmatrix} G_\lambda(x,x) & G_\lambda(x,y) \\ G_\lambda(y,x) & G_\lambda(y,y) \end{pmatrix} = \left[\begin{pmatrix} V(x) & \\ & V(y) \end{pmatrix} - K_{2k \times 2k} \right]^{-1} \ ,$$

where $K_{2k \times 2k}$ is independent of $v(x), v(y)$. Applying the Schur–Banachiewicz formula once again, we obtain:

$$G_\lambda(x,y) = L_{k \times k} (V(y) - M_{k \times k})^{-1} = \frac{L_{k \times k} (V(y) - M_{k \times k})^{\mathrm{Adj}}}{\det(V(y) - M_{k \times k})} \ ,$$

where $L_{k \times k}$ and $M_{k \times k}$ are independent of $v(y)$, and Adj denotes the adjugate (= cofactor) matrix.

$$G_\lambda(x,y)(V(y) - \lambda) = \frac{L_{k \times k} (V(y) - M_{k \times k})^{\mathrm{Adj}} (V(y) - \lambda)}{\det(V(y) - M_{k \times k})} \ .$$

[2] See Henderson and Searle [13] for the history of block matrix inversion formulae

Therefore every entry of $G_\lambda(x,y)(V(y)-\lambda)$ is a ratio of two polynomials Q_1, Q_2 of degree $\leq k$ with respect to the variable $v(y)$. For sufficiently small $s > 0$, Proposition 3.1 implies that for any such pair Q_1, Q_2

$$\left\{ \int \frac{|Q_1(v)|^{2s}}{|Q_2(v)|^{2s}} d\mu(v) \right\}^{1/2} \leq \tilde{C} \int \frac{|Q_1(v)|^s}{|Q_2(v)|^s} d\mu(v) \, .$$

Hence

$$\langle \|G_\lambda(x,y)(V(y)-\lambda)\|^{2s} \rangle_{v(y)}^{1/2} \leq k\tilde{C} \langle \|G_\lambda(x,y)(V(y)-\lambda)\|^s \rangle_{v(y)} \, .$$

Proposition 3.1 also implies that for sufficiently small s

$$\langle \|(V(y)-\lambda)^{-1}\|^{2s} \rangle_{v(y)}^{1/2} \leq 2k(1+|\lambda|)^{-s} \, . \tag{3.2}$$

Indeed, using Proposition 3.1 we first observe that for sufficiently small s

$$\langle \|(v(y)A(y)+B(y)+i)(V(y)-\lambda)^{-1}\|^{2s} \rangle_{v(y)}^{1/2} \leq 1.1k \, ,$$

Using now the resolvent identity

$$(V(y)-\lambda)^{-1} = -(i+\lambda)^{-1} + (i+\lambda)^{-1}(v(y)A(y)+B(y)+i)(V(y)-\lambda)^{-1} \, ,$$

we establish (3.2).

Returning to (3.1), we obtain:

$$\langle \|G_\lambda(x,y)\|^s \rangle_{v(y)} \leq \hat{C} \langle \|G_\lambda(x,y)(V(y)-\lambda)\|^s \rangle_{v(y)} (1+|\lambda|)^{-s} \, . \tag{3.3}$$

To extend this inequality to all $s \leq \frac{\alpha q}{4\alpha+2q}$, we apply Proposition 3.1 once again. Every entry in $G_\lambda(x,y)$ and $G_\lambda(x,y)(V(y)-\lambda)$ is a ratio of two polynomial functions of $v(y)$ whose degree do not exceed k. By Proposition 3.1, the expressions

$$\left\{ \int \frac{|Q_1(v)|^s}{|Q_2(v)|^s} d\mu(v) \right\}^{1/s}$$

are comparable as long as $q \geq \frac{2ks\alpha}{\alpha - ks}$, that is, $s \leq \frac{q\alpha}{kq+2k\alpha}$. Therefore (3.3) remains valid in this range of s. Averaging over $(v(z))_{z \neq y}$, we obtain

$$\langle \|G_\lambda(x,y)\|^s \rangle \leq \hat{C} \langle \|G_\lambda(x,y)(V(y)-\lambda)\|^s \rangle (1+|\lambda|)^{-s} \, .$$

□

Proof of Proposition 3.1.

Lower bound. Choose $R > 0$ so that $\mu[-R, R] \geq 1/2$ (for example, take $R = \max(1, 2C_{\mathbf{A2}})^{1/q}$.) Then

$$\int \frac{\prod_{j=1}^{l} |v - a_j|^s}{\prod_{k=1}^{m} |v - b_k|^r} \, d\mu(v) \geq \int_{-R}^{R} \\ \geq C_1^{-1} \frac{\prod_{|a_j| \geq 2R} (1 + |a_j|)^s}{\prod_{k=1}^{m} (1 + |b_k|)^r} \int_{-R}^{R} \prod_{|a_j| < 2R} |v - a_j|^s d\mu(v) \, .$$

Now, for any $0 < t < 1$, the set $\{\prod |v - a_j| \leq t\}$ can be covered by l intervals of length $Ct^{1/l}$. Therefore, when $t > 0$ is sufficiently small,

$$\mu \left\{ \prod_{|a_j| < 2R} |v - a_j| \leq t \right\} \leq C_2 t^{\alpha/l} \leq 1/4 \, .$$

Then

$$\int_{-R}^{R} \prod_{|a_j| < 2R} |v - a_j|^s d\mu(v) \geq t^s/4 \geq C_3^{-1} \geq C_4^{-1} \prod_{|a_j| < 2R} (1 + |a_j|)^s \, .$$

Upper bound. Let us start with several reductions. First, it is sufficient to consider the case $a_1 = \cdots = a_l = a$, $b_1 = \cdots = b_m = b$. This follows from the Cauchy–Schwarz inequality

$$\int \frac{\prod_{j=1}^{l} |v - a_j|^s}{\prod_{k=1}^{m} |v - b_k|^r} \, d\mu(v) \leq \prod_{j=1}^{l} \prod_{k=1}^{m} \left\{ \int \frac{|v - a_j|^{sl}}{|v - b_k|^{rm}} \, d\mu(v) \right\}^{\frac{1}{lm}} \, .$$

Second,

$$\int \frac{|v - a|^{sl}}{|v - b|^{rm}} \, d\mu(v) \leq C \left\{ \int \frac{|v|^{sl} \, d\mu(v)}{|v - b|^{rm}} + |a|^{sl} \int \frac{d\mu(v)}{|v - b|^{rm}} \right\} \, ,$$

so it is sufficient to consider the case $a = 0$. Third, we can assume that $|b| > 1$, since for $|b| \leq 1$ the regularity condition **A1** implies

$$\int \frac{|v|^{sl} \, d\mu(v)}{|v - b|^{rm}} \leq C \left\{ \int \frac{d\mu(v)}{|v - b|^{rm - sl}} + |b|^{sl} \int \frac{d\mu(v)}{|v - b|^{rm}} \right\} \leq C_5 \leq \frac{C_6}{(1 + |b|)^{rm}} \, .$$

Therefore we need to show that for $|b| > 1$

$$\int \frac{|v|^{sl}\, d\mu(v)}{|v-b|^{rm}} \le C|b|^{-rm} \, .$$

Let us divide the integral into two parts:

$$\int = \int_{\Big||v|-|b|\Big|>|b|/2} + \int_{|b|/2<|v|<3|b|/2} \, .$$

By **A2**, the first integral is at most

$$\left(\frac{2}{|b|}\right)^{rm} \int |v|^{sl} d\mu(v) \le C_6|b|^{-rm} \, .$$

Let us estimate second integral.

$$\begin{aligned}
\int_{|b|/2<|v|<3|b|/2} &\le \left(\frac{3|b|}{2}\right)^{sl} \int_{|b|/2<|v|<3|b|/2} \frac{d\mu(v)}{|v-b|^{rm}} \\
&\le C_7|b|^{sl} \int_0^\infty \mu\left\{|b|/2<|v|<3|b|/2\, ,\ |v-b|<t^{-\frac{1}{rm}}\right\} dt \\
&= C_7|b|^{sl} \left\{ \int_0^{b^{-sl-rm}} + \int_{b^{-sl-rm}}^{b^\gamma} + \int_{b^\gamma}^\infty \right\} \, ,
\end{aligned} \tag{3.4}$$

where $\gamma > 0$ is a number that we shall choose shortly. The first integral in (3.4) is at most b^{-sl-rm}. The second integral is at most

$$b^\gamma \mu\left\{|v| > |b|/2\right\} \le C_8 b^{\gamma-q} \le C_8 b^{-sl-rm}$$

as long as

$$\gamma \le q - sl - rm \, . \tag{3.5}$$

The third integral is at most

$$\int_{b^\gamma}^\infty \mu\left\{|v-b| < t^{-\frac{1}{rm}}\right\} dt \le C_9 \int_{b^\gamma}^\infty t^{-\frac{\alpha}{rm}}\, dt \le C_{10}|b|^{-\gamma(\frac{\alpha}{rm}-1)} \le C_{11}|b|^{-sl-rm}$$

as long as

$$\gamma \ge \frac{sl+rm}{\frac{\alpha}{rm}-1} \, . \tag{3.6}$$

Since $q \ge (sl+rm)\frac{\alpha}{\alpha-rm}$, we can choose γ that satisfies both (3.5) and (3.6); then we obtain the claimed estimate. □

Now we prove Lemma 2.3.

Proof. We shall prove that

$$\langle |G_\lambda(\mathbf{m},\mathbf{n})|^s |V(\mathbf{n})-\lambda|^s \rangle_{\mathcal{B}_\mathbf{n}} \geq \hat{C}^{-1} \langle |G_\lambda(\mathbf{m},\mathbf{n})|^s \rangle_{\mathcal{B}_\mathbf{n}} (1+|\lambda|)^s \, .$$

First,

$$\langle |G_\lambda(\mathbf{m},\mathbf{n})|^{s/2} \rangle^2_{\mathcal{B}_\mathbf{n}} \leq \langle |G_\lambda(\mathbf{m},\mathbf{n})|^s |V(\mathbf{n})-\lambda|^s \rangle_{\mathcal{B}_\mathbf{n}} \langle |V(\mathbf{n})-\lambda|^{-s} \rangle_{\mathcal{B}_\mathbf{n}} \, ,$$

and, as above,

$$\langle |V(\mathbf{n})-\lambda|^{-s} \rangle_{\mathcal{B}_\mathbf{n}} \leq C(1+|\lambda|)^{-s} \, .$$

Therefore it remains to show that

$$\langle |G_\lambda(\mathbf{m},\mathbf{n})|^s \rangle_{\mathcal{B}_\mathbf{n}} \leq C \langle |G_\lambda(\mathbf{m},\mathbf{n})|^{s/2} \rangle^2_{\mathcal{B}_\mathbf{n}} \, . \tag{3.7}$$

For simplicity of notation, let $\mathcal{B}_\mathbf{n} = \{v_1, \cdots, v_J\}$ (here $1 \leq J \leq k$). Cramer's rule (or the Schur–Banachiewicz formula) shows that, as a function of every v_j, $Q(v_1, \cdots, v_J) = G_\lambda(\mathbf{m},\mathbf{n})$ is a ratio of two polynomials of degree at most k. The next multivariate version of Proposition 3.1 concludes the proof. □

Proposition 3.2. *Let Q be a function of J variables $v_1, \cdots, v_J$ which, as a function of every v_j, is a ratio of two polynomials of degree at most k. Then, for any probability measure μ satisfying the assumptions* **A1**, **A2** *and for any $s \leq \frac{q\alpha}{k(q+2\alpha)}$,*

$$\left\{ \int d\mu(v_1) \cdots d\mu(v_J) |Q(v_1, \cdots, v_J)|^s \right\}^{1/s} \leq C_{k,\alpha,q,C_{\mathbf{A1}},C_{\mathbf{A2}}} \left\{ \int d\mu(v_1) \cdots d\mu(v_J) |Q(v_1, \cdots, v_J)|^{s/2} \right\}^{2/s} .$$

Proof. If $J = 1$, (3.7) follows from Proposition 3.1. Then we proceed by induc-

tion on J. By case $J = 1$,

$$\begin{aligned}
&\int d\mu(v_1)\cdots d\mu(v_J)|Q(v_1,\cdots,V_J)|^s \\
&\quad\le C_1\int d\mu(v_1)\cdots d\mu(v_{J-1})\left\{\int d\mu(v_J)|Q(v_1,\cdots,v_J)|^{s/2}\right\}^2 \\
&\quad= C_1\int d\mu(v_1)\cdots d\mu(v_{J-1}) \\
&\qquad\int d\mu(v_J)|Q(v_1,\cdots,v_J)|^{s/2}\int d\mu(v'_J)|Q(v_1,\cdots,v'_J)|^{s/2} \\
&\quad= C_1\int d\mu(v_J)d\mu(v'_J)\int d\mu(v_1)\cdots d\mu(v_{J-1}) \\
&\qquad|Q(v_1,\cdots,v_J)|^{s/2}|Q(v_1,\cdots,v'_J)|^{s/2}
\end{aligned}$$

By the Cauchy–Schwarz inequality and the induction step, the last expression is at most

$$\begin{aligned}
&C_1\int d\mu(v_J)d\mu(v'_J) \\
&\quad\left\{\int d\mu(v_1)\cdots d\mu(v_{J-1})|Q(v_1,\cdots,v_J)|^s\int d\mu(v'_1)\cdots d\mu(v'_{J-1})|Q(v'_1,\cdots,v'_J)|^s\right\}^{1/2} \\
&\le C_2\int d\mu(v_J)d\mu(v'_J) \\
&\quad\int d\mu(v_1)\cdots d\mu(v_{J-1})|Q(v_1,\cdots,v_J)|^{s/2}\int d\mu(v'_1)\cdots d\mu(v'_{J-1})|Q(v'_1,\cdots,v'_J)|^{s/2} \\
&\qquad= C_2\left\{\int d\mu(v_1)\cdots d\mu(v_J)|Q(v_1,\cdots,v_J)|^{s/2}\right\}^2 .
\end{aligned}$$

□

4 Dynamical localisation

The proof of dynamical localisation is based on the notion of eigenfunction correlators, introduced by Aizenman in [1].

Let us start with some definitions, which are adjusted from the lecture notes of Aizenman and Warzel [3] to our block setting. Let H be an operator acting on

$\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^k$. For $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, the (matrix-valued) spectral measure $\mu_{\mathbf{mn}}$ is defined by

$$\int \phi \, d\mu_{\mathbf{mn}} = \phi(H)(\mathbf{m}, \mathbf{n}) \, , \quad \phi \in C_0(\mathbb{R}) \, .$$

The eigenfunction correlator $Q_I(\mathbf{m}, \mathbf{n})$ corresponding to a finite interval $I \subset \mathbb{R}$ (on the energy axis) is defined by

$$Q_I(\mathbf{m}, \mathbf{n}) = \sup \{ \|\phi(H)(\mathbf{m}, \mathbf{n})\| \mid \operatorname{supp} \phi \subset I, \, |\phi| \leq 1 \} \, .$$

Obviously,

$$\sup_{t \geq 0} |e^{itH_I}(\mathbf{m}, \mathbf{n})| \leq Q_I(\mathbf{m}, \mathbf{n}) \tag{4.1}$$

for any $t > 0$.

The eigenfunction correlators can be also defined for the restriction of H to a finite box Λ (we denote this restriction by the superscript Λ). In this case, it satisfies the following inequalities (the first one is an equality in the scalar case, cf. [3]):

Lemma 4.1.

$$Q_I^\Lambda(\mathbf{m}, \mathbf{n}) \leq \lim_{\epsilon \to +0} \frac{\epsilon}{2} \int_I \|G^\Lambda_{\lambda + i0}(\mathbf{m}, \mathbf{n})\|^{1-\epsilon} d\lambda \leq k \, .$$

Proof. For any eigenvalue ν of H^Λ, define a $k \times k$ matrix

$$M_\nu = \sum \psi(\mathbf{m}) \otimes \psi(\mathbf{n}) : u \mapsto \sum (\psi(\mathbf{n}) \cdot u) \psi(\mathbf{m}) \, ,$$

where the sum is over all eigenfunctions ψ of H^Λ corresponding to ν. Then

$$\phi(H^\Lambda)(\mathbf{m}, \mathbf{n}) = \sum_{\nu \in I} \phi(\nu) M_\nu \, ,$$

whereas

$$G^\Lambda_\lambda(\mathbf{m}, \mathbf{n}) = \sum_\nu \frac{M_\nu}{\nu - \lambda}$$

(where now the sum is over all eigenvalues of H^Λ.) Therefore

$$\|\phi(H^\Lambda)(\mathbf{m}, \mathbf{n})\| \leq \sum_{\nu \in I} \|M_\nu\| = \lim_{\epsilon \to +0} \frac{\epsilon}{2} \int_I \|G^\Lambda_\lambda(\mathbf{m}, \mathbf{n})\|^{1-\epsilon} d\lambda \, .$$

The equality can be proved by representing $I = \uplus I_\nu$ as a disjoint union of neighbourhoods of $\nu \in I$, and noting than

$$G^\Lambda_\lambda(\mathbf{m},\mathbf{n}) = \frac{M_\nu}{\nu-\lambda} + O(1), \quad \lambda \to \nu \ .$$

Also,

$$\sum_{\nu\in I} \|M_\nu\| \le \sum_\psi \|\psi(\mathbf{m}) \otimes \psi(\mathbf{n})\| = \sum_\psi \|\psi(\mathbf{m})\|\|\psi(\mathbf{n})\|$$
$$\le \left\{ \sum_\psi \|\psi(\mathbf{m})\|^2 \sum_\psi \|\psi(\mathbf{n})\|^2 \right\}^{1/2} = k \ .$$

□

Now assume that $H = H_\omega$ is a random operator of the form (1.2), where the random $V(\mathbf{m})$ are independent and identically distributed, $K(\mathbf{m},\mathbf{n})$ depends only on $\mathbf{m}-\mathbf{n}$. We shall prove

Theorem 4.2. *Let $0 < s < 1$, and suppose for every box Λ, every $\lambda \in I$, and every $\mathbf{m},\mathbf{n} \in \Lambda$*

$$\langle \|G^\Lambda_{\lambda+i0}(\mathbf{m},\mathbf{n})\|^s \rangle \le C \exp(-\gamma \operatorname{dist}(\mathbf{m},\mathbf{n}))$$

for some $C,\gamma > 0$, where G^Λ is the resolvent of the restriction of H to Λ. Then, for every $\mathbf{m},\mathbf{n} \in \mathbb{Z}^d$,

$$\langle Q_I(\mathbf{m},\mathbf{n}) \rangle \le C' \operatorname{dist}^{2d}(\mathbf{m},\mathbf{n}) \exp(-\frac{s\gamma}{8} \operatorname{dist}(\mathbf{m},\mathbf{n})) \ .$$

Remark. *A similar statement can be proved for potentials of the form (1.1).*

Proof. We shall prove the estimate in a large box Λ containing $\mathbf{m},\mathbf{n}$ (uniformly in the size of Λ). Let $\Lambda_\mathbf{m}$ and $\Lambda_\mathbf{n}$ be two boxes of radius $R = \lfloor \operatorname{dist}(\mathbf{m},\mathbf{n})/2 \rfloor$, centered at $\mathbf{m}$,$\mathbf{n}$, respectively. According to the resolvent identity,

$$G^\Lambda_{\lambda+i0}(\mathbf{m},\mathbf{n}) = g^{-1} \sum_{\mathbf{w}\mathbf{w}'\in\partial\Lambda_\mathbf{m}} G^{\Lambda_\mathbf{m}}_{\lambda+i0}(\mathbf{m},\mathbf{w}) K(\mathbf{w},\mathbf{w}') G^\Lambda_{\lambda+i0}(\mathbf{w}',\mathbf{n}) \ ,$$

where the sum is over all pairs $\mathbf{w}\mathbf{w}'$ such that $\mathbf{w} \in \Lambda_\mathbf{m}$, $\mathbf{w}' \notin \Lambda_\mathbf{m}$, $\mathbf{w} \sim \mathbf{w}'$. Therefore

$$\|G^\Lambda_{\lambda+i0}(\mathbf{m},\mathbf{n})\| \le C g^{-1} \max_{\mathbf{w}\mathbf{w}'\in\partial\Lambda_\mathbf{m}} \|G^{\Lambda_\mathbf{m}}_{\lambda+i0}(\mathbf{m},\mathbf{w})\| \sum_{\mathbf{w}\mathbf{w}'\in\partial\Lambda_\mathbf{m}} \|G^\Lambda_{\lambda+i0}(\mathbf{w}',\mathbf{n})\| \ .$$

Now we apply [8, Prop. 5.1] (which holds in the block-operator setting). It shows that, with probability at least $1 - C'R^{2d}\exp(-\gamma sR/8)$, one can decompose $I = I_{\mathbf{m}} \cup I_{\mathbf{n}}$ so that for every $\mathbf{w}\mathbf{w}' \in \partial\Lambda_{\mathbf{m}}$ and $\lambda \in I_{\mathbf{m}}$

$$\max_{\mathbf{w}\mathbf{w}'\in\partial\Lambda_{\mathbf{m}}} \|G^{\Lambda_{\mathbf{m}}}_{\lambda+i0}(\mathbf{m},\mathbf{w})\| \leq C\exp(-\gamma R/8)\ ,$$

and for every $\mathbf{w}\mathbf{w}' \in \partial\Lambda_{\mathbf{n}}$ and $\lambda \in I_{\mathbf{n}}$

$$\max_{\mathbf{w}\mathbf{w}'\in\partial\Lambda_{\mathbf{n}}} \|G^{\Lambda_{\mathbf{n}}}_{\lambda+i0}(\mathbf{n},\mathbf{w})\| \leq C\exp(-\gamma R/8)\ .$$

Therefore,

$$\lim_{\epsilon\to+0}\frac{\epsilon}{2}\int_{I_{\mathbf{m}}} \|G^{\Lambda}_{\lambda+i0}(\mathbf{m},\mathbf{n})\|^{1-\epsilon}d\lambda \leq C''g^{-1}\exp(-R\gamma/8)R^{d-1}\ ,$$

and the same estimate holds for the integral over $I_{\mathbf{n}}$. Therefore finally

$$\begin{aligned}\langle Q_I(\mathbf{m},\mathbf{n})\rangle &\leq C^{IV}g^{-1}R^{d-1}\exp(-R\gamma/8) + C'kR^{2d}\exp(-\gamma sR/8)\\ &\leq C^{V}R^{2d}\exp(-\gamma sR/8)\ .\end{aligned}$$

□

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